

Coeffective basic cohomologies of K -contact and Sasakian manifolds

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Abstract

In this paper we define coeffective de Rham cohomology for basic forms on a K -contact or Sasakian manifold M and we discuss its relation with usually basic cohomology of M . When M is of finite type (for instance it is compact) several inequalities relating some basic coeffective numbers to classical basic Betti numbers of M are obtained. In the case of Sasakian manifolds, we define and study coeffective Dolbeault and Bott-Chern cohomologies for basic forms. Also, in this case, we prove some Hodge decomposition theorems for coeffective basic de Rham cohomology, relating this cohomology with coeffective basic Dolbeault or Bott-Chern cohomology. The notions are introduced in a similar manner with the case of symplectic and Kähler manifolds.

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1 Introduction and preliminary notions

1.1 Introduction

The coeffective cohomology was introduced by Bouché in [6] for symplectic manifolds. More exactly, a symplectic form ω defines a special subcomplex of the de Rham complex $(F^\bullet(M), d)$ of differential forms on M : it consists of those forms φ which are annihilated by ω , that is, $\varphi \wedge \omega = 0$. Since ω is closed, we have in fact a subcomplex of $(F^\bullet(M), d)$ whose cohomology is called coeffective. This cohomology is related with the truncated de Rham cohomology by the class ω . Further significant developments of coeffective cohomologies in many different contexts (symplectic, Kähler, (almost) cosymplectic, (almost) contact, quaternionic manifolds) are given by a series of papers of de Andrés, Fernández, de León, Ibáñez, Mencía, Chinea, Marrero [2, 8, 9, 14, 15, 16] and others papers by these authors. For Kähler manifolds both cohomologies (coeffective cohomology and de Rham cohomology truncated by $[\omega]$) are isomorphic for $p \neq n$, $\dim M = 2n$, though in general they are different for non Kähler symplectic manifolds [2]. For symplectic manifolds of finite type was introduced the coeffective numbers of the symplectic manifold and several inequalities relating them to the Betti numbers. Similar results were obtained in the context of almost contact [9, 15] and quaternionic manifolds [16]. Also, a coeffective Dolbeault cohomology for compact Kähler and indefinite Kähler manifolds is studied in [18].

Our aim in this paper is to study the coeffective de Rham, Dolbeault and Bott-Chern cohomologies for basic forms of (compact) K -contact or Sasakian manifolds with respect to the Reeb

foliation \mathcal{F}_ξ of the fundamental Reeb vector field ξ , giving a new contribution concerning to basic cohomology of K -contact or Sasakian manifolds. Notice that a background about the basic cohomology of K -contact and Sasakian manifolds can be found in the Ch. VII from [7]. Other developments of basic cohomologies of Sasakian manifolds in a similar direction as in the recent studies for symplectic manifolds, [27], are given in [19].

The structure of paper is the following:

In the preliminary subsection, following [7, 23] we briefly recall some elementary definitions about basic forms, basic star operator, basic de Rham Laplacian, basic de Rham cohomology on K -contact manifolds.

We notice that if η is the contact form of M then $d\eta$ is basic with respect to the Reeb foliation \mathcal{F}_ξ . Thus, in the Section 2 we begin our study with the coeffective de Rham cohomology for basic forms on a K -contact manifold M . The main ingredient is given by the isomorphism between the space of basic differential forms on M and the space of differential forms on the orbit space M_ξ of \mathcal{F}_ξ , which is known that it is symplectic (or Kählerian in the Sasakian case). Thus, following the classical study of coeffective cohomology of symplectic manifolds, see [14], we define the coeffective basic de Rham cohomology of M and we prove that when M is compact Sasakian

$$H^p(\mathcal{A}_b(M)) \cong \tilde{H}_b^p(M), \quad \forall p \neq n,$$

where $H^p(\mathcal{A}_b(M))$ denotes the coeffective basic cohomology group of degree p of M and $\tilde{H}_b^p(M)$ is the subspace of the basic de Rham cohomology group of M consisting of those classes $a \in H_b^p(M)$ such that $a \wedge [d\eta] = 0$, or in other words, the truncated basic de Rham cohomology group of degree p . Notice that for an arbitrary K -contact manifold $H^p(\mathcal{A}_b(M))$ vanishes for every $p \leq n - 1$, where $\dim M = 2n + 1$. Also, using a technique based on the long exact sequence in cohomology associated with an exact short sequence of complexes, we obtain that the coeffective basic de Rham cohomology groups of a K -contact or Sasakian manifold M of finite type have finite dimension. In this case, if we denote by $c_b^p(M) = \dim H^p(\mathcal{A}_b(M))$, called the coeffective basic numbers of order p of M , then they satisfy the following inequalities:

$$b_b^p(M) - b_b^{p+2}(M) \leq c_b^p(M) \leq b_b^p(M) + b_b^{p+1}(M), \quad \forall p \geq n + 1,$$

where $b_b^p(M) = \dim H_b^p(M)$ is the basic Betti number of order p of M .

As a consequence, for a compact Sasakian manifolds, we deduce that

$$c_b^p(M) = b_b^p(M) - b_b^{p+2}(M), \quad \forall p \geq n + 1,$$

which means that the coeffective basic numbers of a compact Sasakian manifold measure the jumps between the basic Betti numbers.

In the end of Section 2, using an exact sequence in cohomology (which is the foliation analogue of the Gysin sequence), we prove that the following isomorphism holds:

$$H^p(\mathcal{A}(M)) \cong H^{p-1}(\mathcal{A}_b(M)), \quad \forall p = 1, \dots, 2n + 1,$$

where $H^\bullet(\mathcal{A}(M))$ is the coeffective de Rham cohomology of M considered in [15].

In Section 3 we consider basic forms with complex valued on a Sasakian manifold M and taking into account that $d\eta$ is a basic form of complex type $(1, 1)$, in a similar manner with coeffective Dolbeault cohomology of Kähler manifolds, see [18], we define and study coeffective basic Dolbeault cohomology of a Sasakian manifold. We prove that when M is compact

$$H^{r,s}(\mathcal{A}_b(M)) \cong \tilde{H}_b^{r,s}(M), \quad \forall r + s \neq n,$$

where $H^{r,s}(\mathcal{A}_b(M))$ denotes the coeffective basic cohomology group of type (r, s) of M and $\tilde{H}_b^{r,s}(M)$ is the truncated basic Dolbeault cohomology group of type (r, s) . In the case when M is a compact Sasakian manifold, we prove a Hodge decomposition theorem for coeffective basic de Rham cohomology of M , relating this cohomology with coeffective basic Dolbeault cohomology of M . Also, several inequalities relating the coeffective basic Hodge numbers to the classical basic Hodge numbers are given similarly as for in the de Rham case.

The aim of Section 4 is to construct a coeffective basic Bott-Chern cohomology of a Sasakian manifold M . In this sense we firstly define basic Bott-Chern and Aeppli cohomology of M and we obtain a Hodge-Bott-Chern decomposition theorem for basic forms of M . Next, in similar manner with the study of coeffective basic de Rham and Dolbeault cohomology of M , we define and study a coeffective Bott-Chern cohomology for basic forms on M .

The main methods used here are similarly and closely related to those used in [6, 9, 14, 15, 16].

1.2 Preliminaries

Let (M, F, ξ, η, g) be a $(2n+1)$ -dimensional *almost contact manifold*; that is see [3, 4, 7, 23], F is an $(1, 1)$ -tensor field, η is an 1-form, ξ is a vector field, and g is a Riemannian metric on M such that

$$F^2 = -\text{Id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \text{and} \quad g(FX, FY) = g(X, Y) - \eta(X)\eta(Y) \quad (1.1)$$

for every $X, Y \in \mathcal{X}(M)$, where Id is the identity transformation. Then we have $F(\xi) = 0$ and $\eta(X) = g(X, \xi)$ for all $X \in \mathcal{X}(M)$. The *fundamental 2-form* Φ of M is defined by $\Phi(X, Y) = g(X, FY)$, and the $(2n+1)$ -form $\eta \wedge \Phi^n$ is a volume form on M . The almost contact metric manifold is said to be: *contact* if $d\eta = \Phi$; *K-contact* if it is contact and ξ is Killing; *normal* if $[F, F] + 2d\eta \otimes \xi = 0$; *Sasakian* if it is contact and normal. If M is Sasakian manifold then it is *K-contact* [3].

Consider the field $F^0(M) = \mathcal{F}(M)$ of smooth real valued functions defined on M . For each $p = 1, \dots, 2n+1$ denote by $F^p(M)$ the module of p -forms, by $F(M) = \oplus_{p \geq 0} F^p(M)$ the exterior algebra of M , and by \langle, \rangle , the natural scalar product on $F(M)$.

Recall that the differential form ω on M is called *basic* if it is *horizontal* (that is $\iota_\xi \omega = 0$, where ι_ξ denotes the interior product with respect to ξ) and *invariant* (that is $\mathcal{L}_\xi \omega = 0$, where \mathcal{L}_ξ denotes the Lie derivative with respect to ξ). Denote by $F_b^p(M)$ the subspace of all basic p -forms on the manifold M . It is a module over the ring $F_b^0(M) = \mathcal{F}_b(M)$ of *basic functions* on M (that is, $\mathcal{L}_\xi f = 0$) and let $F_b(M) = \oplus_{p \geq 0} F_b^p(M)$ be the graded algebra of all basic forms on M . By Cartan identity $\mathcal{L}_\xi = d\iota_\xi + \iota_\xi d$, we easily obtain that the exterior differential of a basic form is also basic, so we can consider the basic differential $d_b = d|_{F_b^p(M)} : F_b^p(M) \rightarrow F_b^{p+1}(M)$.

Thus, the basic forms constitute a subcomplex $(\oplus_{p \geq 0} F_b^p(M), d_b)$ of the de Rham complex $(\oplus_{p \geq 0} F^p(M), d)$. The cohomology of this subcomplex is defined by

$$H_b(M) = \oplus_{p \geq 0} H_b^p(M), \quad H_b^p(M) = \ker\{d_b : F_b^p(M) \rightarrow F_b^{p+1}(M)\} / d_b(F_b^{p-1}(M)).$$

This cohomology play the role of de Rham cohomology of the orbit space of the *K-contact* manifold M and we call it the *basic de Rham cohomology* or simply the *basic cohomology* of M . Moreover, the space of basic cohomology $H_b^\bullet(M)$ is an invariant of the characteristic foliation \mathcal{F}_ξ and therefore is an invariant of the *K-contact* structure on the manifold M . The relation between the basic cohomology $H_b^\bullet(M)$ and de Rham cohomology $H^\bullet(M)$ of the *K-contact* manifold M is the same as in the general case of a foliation generated by a nonsingular Killing vector field (see for

instance [26], Theorem 10.13, pg. 139). On compact K -contact manifolds the basic cohomology groups enjoy some special properties. In particular, there is a transverse Hodge theory [7, 11, 13].

Let \star be the usual star operator on M . If $\omega \in F_b^p(M)$ then the $(2n-p)$ -form $\iota_\xi \star \omega$ is basic. Therefore we can define the *basic star operator* $\star_b : F_b^p(M) \rightarrow F_b^{2n-p}(M)$ by

$$\star_b \omega = (-1)^p \iota_\xi \star \omega. \quad (1.2)$$

Also, the usual scalar product \langle, \rangle on $F^p(M)$ restricted to basic forms, is denoted by \langle, \rangle_b , and it is given by

$$\langle \omega, \theta \rangle_b = \int_M \omega \wedge \star_b \theta \wedge \eta, \quad (1.3)$$

for all $\omega, \theta \in F_b^p(M)$ and we denote by the symbol A^* the adjoint of the operator $A : F_b(M) \rightarrow F_b(M)$ with respect to \langle, \rangle_b . As it is well known, the \langle, \rangle_b -adjoint d_b^* of d_b satisfies $d_b^* = -\star_b d_b \star_b$.

The *basic de Rham Laplacian* Δ_b is defined in terms of d_b and its adjoint d_b^* by

$$\Delta_b = d_b d_b^* + d_b^* d_b. \quad (1.4)$$

The space $\mathcal{H}_b^p(M)$ of *basic harmonic p -forms* on M is then defined to be the kernel of $\Delta_b : F_b^p(M) \rightarrow F_b^p(M)$, and $\mathcal{H}_b^p(M) = \ker d_b \cap \ker d_b^*$. The transverse Hodge theorem [13] then says that

$$F_b^p(M) = \text{Im } d_b \oplus \text{Im } d_b^* \oplus \ker \Delta_b \quad (1.5)$$

see also [10], and each basic cohomology class has an unique harmonic representative, i.e.

$$H_b^p(M) \cong \mathcal{H}_b^p(M). \quad (1.6)$$

2 Coeffective de Rham cohomology for basic forms

Throughout this section M is a (compact) K -contact manifold of dimension $2n+1$ and sometimes M is Sasakian. We start with a fundamental result which play an important role for our purpose.

Theorem 2.1. ([23]) *Let M be a K -contact manifold of dimension $2n+1$ and M_ξ the orbit space of the Reeb foliation \mathcal{F}_ξ defined by ξ . If $\pi : M \rightarrow M_\xi$ is the natural projection then $\pi^* : F^p(M_\xi) \rightarrow F_b^p(M)$ is an isomorphism.*

Proof. Obviously, π^* is injective.

We prove now that for any $\varphi \in F_b^p(M)$ there exists $\varphi' \in F^p(M_\xi)$ such that $\varphi = \pi^* \varphi'$. Since φ is horizontal (that is $\iota_\xi \varphi = 0$), the values $\varphi(X_1, \dots, X_p)$ can be non zero only when the tangent vectors $\{X_1, \dots, X_p\} \in T_x M$ are orthogonal on ξ . But the condition $\mathcal{L}_\xi \varphi = 0$ shows that φ is invariant by the Reeb group $\{\Phi_t\}_{t \in \mathbb{R}}$, that is $\Phi_t^* \varphi = \varphi$. It follows that

$$\varphi(\Phi_{t*} X_1, \dots, \Phi_{t*} X_p) = \varphi(X_1, \dots, X_p)$$

and so at the point $\pi(x)$ is well-defined a p -form $\varphi'_{\pi(x)}$ with the property $\varphi_x = \pi^* \varphi'_{\pi(x)}$. But x is arbitrary in M and then is well-defined the p -form $\varphi' \in F^p(M_\xi)$ with the property $\varphi = \pi^* \varphi'$, which proves that $F_b^p(M) \subseteq \text{Im } \pi^*$.

It remains only to prove that $\text{Im } \pi^* \subseteq F_b^p(M)$. Remark that for any $\varphi' \in F^p(M_\xi)$ we have $\Phi_t^* \pi^* \varphi' = \pi^* \varphi'$, $\iota_\xi \pi^* \varphi' = 0$, hence $\mathcal{L}_\xi \pi^* \varphi' = 0$ and then $\pi^* \varphi' \in F_b^p(M)$. \square

Also, it is well known that (M_ξ, Ω) is symplectic with $d\eta = \pi^*\Omega$, and when M is Sasakian then (M_ξ, Ω) is Kählerian. We have now

Lemma 2.1. *The operator $L : F_b^p(M) \rightarrow F_b^{p+2}(M)$ defined by $L\varphi = \varphi \wedge d\eta$ is injective for $p \leq n-1$ and surjective for $p \geq n-1$.*

Proof. According to [6], we have that the symplectic operator $L' : F^p(M_\xi) \rightarrow F^{p+2}(M_\xi)$ given by $L'\varphi' = \varphi' \wedge \Omega$ is surjective for $p \geq n-1$ and injective for $p \leq n-1$. Now, by Theorem 2.1, for every $\varphi \in F_b^p(M)$ there is $\varphi' \in F^p(M_\xi)$ such that $\varphi = \pi^*\varphi'$ and

$$L\varphi = \varphi \wedge d\eta = \pi^*\varphi' \wedge \pi^*\Omega = \pi^*(\varphi' \wedge \Omega) = (\pi^* \circ L')\varphi'.$$

Thus, the operator L is also injective for $p \leq n-1$ and surjective for $p \geq n-1$. \square

Now, as in the case of classical coeffective cohomology, see [6, 9, 15], we consider the subspace $\mathcal{A}_b^p(M) \subset F_b^p(M)$ defined by

$$\mathcal{A}_b^p(M) = \{\varphi \in F_b^p(M) \mid \varphi \wedge d\eta = 0\} = \ker L|_{F_b^p(M)}.$$

A basic form $\varphi \in \mathcal{A}_b^p(M)$ is said to be a *coeffective basic p -form* on M .

Since d_b commutes with L , we can consider the subcomplex of basic de Rham complex of M , namely $(\mathcal{A}_b^\bullet(M), d_b)$, called the *coeffective basic de Rham complex* of M . The cohomology groups of this complex are called *coeffective basic de Rham cohomology groups* of M and they are denoted by $H^p(\mathcal{A}_b(M))$.

As a consequence of Lemma 2.1, one gets

Proposition 2.1. *Let M be a K -contact manifold of dimension $2n+1$. Then $\mathcal{A}_b^p(M) = \{0\}$ for $p \leq n-1$, therefore*

$$H^p(\mathcal{A}_b(M)) = \{0\}, \text{ for } p \leq n-1. \quad (2.1)$$

Let us consider the subspace of $\tilde{H}_b^p(M)$ given by the basic de Rham cohomology classes truncated by the basic de Rham class $[d\eta]$, namely,

$$\tilde{H}_b^p(M) = \{a \in H_b^p(M) \mid a \wedge [d\eta] = 0\}. \quad (2.2)$$

We notice that as in the case of compact cosymplectic manifolds or compact Kähler manifolds, see [9, 18], we can obtain a relation between the coeffective basic de Rham cohomology of a compact Sasakian manifold M and the basic de Rham cohomology of M truncated by $[d\eta]$ in the following way.

Let us denote by $[\cdot]$ the basic de Rham cohomology classes and by $\{\cdot\}$ the coeffective basic de Rham classes.

Proposition 2.2. *For any K -contact manifold M of dimension $2n+1$ the natural mapping*

$$\alpha_p(\{\varphi\}) = [\varphi], \quad (2.3)$$

is surjective for $p \geq n$.

Proof. Let $a \in \tilde{H}_b^p(M)$, that is, $a \in H_b^p(M)$ and $a \wedge [d\eta] = 0$ in $H_b^{p+2}(M)$. Consider a representative φ of a and suppose that $\varphi \notin \mathcal{A}_b^p(M)$ (notice that if $\varphi \in \mathcal{A}_b^p(M)$, then φ defines a basic cohomology class in $H^p(\mathcal{A}_b(M))$ such that $\alpha(\{\varphi\}) = a$).

Since $a \wedge [d\eta] = 0$, then there exists $\sigma \in F_b^{p+1}(M)$ such that $\varphi \wedge d\eta = d_b\sigma$. Then, from Lemma 2.1, there exists $\gamma \in F_b^{p-1}(M)$ such that $L\gamma = \sigma$. Thus, $L(\varphi - d_b\gamma) = 0$ and $d_b(\varphi - d_b\gamma) = 0$. Hence, $\varphi - d_b\gamma$ defines a basic cohomology class in $H^p(\mathcal{A}_b(M))$ such that $\alpha_p(\{\varphi - d_b\gamma\}) = a$. \square

We also notice that for compact Sasakian manifolds, we have

$$\Delta_b L = L \Delta_b. \quad (2.4)$$

This follows by a direct calculation using the following known identities from Sasakian geometry [17, 22, 23], namely

$$\begin{aligned} \Delta \varphi &= \Delta_b \varphi + L \Lambda \varphi + e_\eta \Lambda d \varphi - e_\eta d \Lambda \varphi, \\ \Delta L \varphi - L \Delta \varphi &= 4(n - p - 1) L \varphi + 4 d e_\eta \varphi, \end{aligned}$$

and

$$(\Lambda L^k - L^k \Lambda) \varphi = 4k[(n - p - k + 1) L^{k-1} \varphi + e_\eta \iota_\xi L^{k-1} \varphi],$$

where $L^0 \varphi = \varphi$ and $L^{-1} \varphi = 0$. Here $\Delta = dd^* + d^*d$ is the usual Laplacian on M , $e_\eta : F^p(M) \rightarrow F^p(M)$ is defined by $e_\eta \varphi = \eta \wedge \varphi$ and $\Lambda = \star L \star = -\star_b L \star_b$ is the adjoint of L with respect to \langle, \rangle and \langle, \rangle_b , respectively.

The relation (2.4) say that the map $L : \mathcal{H}_b^p(M) \rightarrow \mathcal{H}_b^{p+2}(M)$ is well defined on the space of harmonic basic p -forms on M . Moreover, by Bouché result for compact Kähler manifolds, see [6], and taking into account the Theorem 2.1, we have that

Lemma 2.2. *Let M be a compact Sasakian manifold of dimension $2n + 1$. The operator $L : \mathcal{H}_b^p(M) \rightarrow \mathcal{H}_b^{p+2}(M)$ is surjective for $p \geq n - 1$.*

This, also follows directly using Lemma 2.1 and basic Hodge decomposition (1.5). We have

Theorem 2.2. *Let M be a compact Sasakian manifold of dimension $2n + 1$. Then*

$$H^p(\mathcal{A}_b(M)) \cong \tilde{H}_b^p(M), \quad \forall p \neq n. \quad (2.5)$$

Proof. The proof it follows by two cases.

Case 1: $p \leq n - 1$.

From (2.1) we know that $H^p(\mathcal{A}_b(M)) = \{0\}$ for $p \leq n - 1$. Moreover, from the isomorphism (1.6), we have

$$\tilde{H}_b^p(M) \cong \left\{ \varphi \in \mathcal{H}_b^p(M) \mid \varphi \wedge d\eta \in d_b \left(F_b^{p+1}(M) \right) \right\} \cong \left\{ \varphi \in \mathcal{H}_b^p(M) \mid \varphi \wedge d\eta = 0 \right\}. \quad (2.6)$$

Thus, from Lemma 2.1 we conclude that $\tilde{H}_b^p(M) = \{0\}$ for $p \leq n - 1$. This finishes the proof for $p \leq n - 1$.

Case 2: $p \geq n + 1$.

We shall see that the mapping α_p given by (2.3) is an isomorphism for $p \geq n + 1$. From Proposition 2.2, it is sufficient to show the injection.

Let $a \in H^p(\mathcal{A}_b(M))$ such that $\alpha_p(a) = 0$ in $\tilde{H}_b^p(M)$ and suppose that φ is a representative of a . Since $\alpha_p(a) = \alpha_p(\{\varphi\}) = [\varphi] = 0$ in $\tilde{H}_b^p(M)$, there exists $\psi \in F_b^{p-1}(M)$ such that

$$\varphi = d_b \psi.$$

Suppose $\psi \notin \mathcal{A}_b^{p-1}(M)$ (notice that if $\psi \in \mathcal{A}_b^{p-1}(M)$, then $a = 0$ and we conclude the proof). Since L commute with d_b , then $d_b(L\psi) = L(d_b\psi) = L\varphi = 0$; therefore $L\psi$ defines a basic cohomology class $[L\psi] \in H_b^{p+1}(M)$. From the isomorphism (1.6), we have

$$L\psi = h + d_b \gamma,$$

for $h \in \mathcal{H}_b^{p+1}(M)$, $\gamma \in F_b^p(M)$. Since $p \geq n+1$ and $h \in \mathcal{H}_b^{p+1}(M)$ by Lemma 2.2 there exists $\sigma \in \mathcal{H}_b^{p-1}(M)$ such that $L\sigma = h$ and since $p-1 \geq n$ by Lemma 2.1 there exists $\sigma_1 \in F_b^{p-2}(M)$ such that $\gamma = L\sigma_1$. Thus,

$$L(\psi - \sigma - d_b\sigma_1) = 0 \text{ and } d_b(\psi - \sigma - d_b\sigma_1) = \varphi.$$

Then, $a = \{\varphi\}$ is the basic zero class in $H^p(\mathcal{A}_b(M))$ and this finishes the proof. \square

Now following an argument similar that in [15], we relate the coeffective basic de Rham cohomology with the basic de Rham cohomology of K -contact manifolds by means of a long exact sequence in basic cohomology.

Let us consider the following short exact sequence for any degree p :

$$0 \longrightarrow \ker L|_{F_b^p(M)} = \mathcal{A}_b^p(M) \xrightarrow{i_b} F_b^p(M) \xrightarrow{L} \text{Im}_b^{p+2} L \longrightarrow 0. \quad (2.7)$$

Since L commutes with d_b , the sequence (2.7) becomes a short exact sequence of basic differential complexes:

$$0 \longrightarrow (\ker L|_{F_b^p(M)}, d_b) = (\mathcal{A}_b^p(M), d_b) \xrightarrow{i_b} (F_b^p(M), d_b) \xrightarrow{L} (\text{Im}_b^{p+2} L, d_b) \longrightarrow 0. \quad (2.8)$$

Therefore, we have the associated long exact sequence in cohomology [28]:

$$\dots \longrightarrow H^p(\mathcal{A}_b(M)) \xrightarrow{H(i_b)} H_b^p(M) \xrightarrow{H(L)} H^{p+2}(\text{Im}_b L) \xrightarrow{\delta_{p+2}^b} H^{p+1}(\mathcal{A}_b(M)) \longrightarrow \dots, \quad (2.9)$$

where $H(i_b)$ and $H(L)$ are the induced homomorphisms in basic cohomology by i_b and L , respectively, and δ_{p+2}^b is the connecting homomorphism defined in the following way: for $[\varphi] \in H^{p+2}(\text{Im}_b L)$, then $\delta_{p+2}^b[\varphi] = [d_b\psi]$, for $\psi \in F_b^p(M)$ such that $L\psi = \varphi$.

From Lemma 2.1 it results that $\text{Im}_b^{p+2} L = F_b^{p+2}(M)$, for $p \geq n-1$. As a consequence, we have

$$H^{p+2}(\text{Im}_b L) = H_b^{p+2}(M), \quad \forall p \geq n.$$

Furthermore, the long exact sequence in basic cohomology (2.9) may be expressed as

$$\dots \longrightarrow H^p(\mathcal{A}_b(M)) \xrightarrow{H(i_b)} H_b^p(M) \xrightarrow{H(L)} H_b^{p+2}(M) \xrightarrow{\delta_{p+2}^b} H^{p+1}(\mathcal{A}_b(M)) \longrightarrow \dots \quad (2.10)$$

for $p \geq n$. Now, we shall decompose the long exact sequence (2.10) in 5 terms exact sequences:

$$0 \rightarrow \text{Im } \delta_{p+1}^b = \ker H(i_b) \xrightarrow{i} H^p(\mathcal{A}_b(M)) \xrightarrow{H(i_b)} H_b^p(M) \xrightarrow{H(L)} H_b^{p+2}(M) \xrightarrow{\delta_{p+2}^b} \text{Im } \delta_{p+2}^b \rightarrow 0. \quad (2.11)$$

If $H_b^p(M)$ are finite dimensional (for instance if M is compact) we denote by $b_b^p(M) = \dim H_b^p(M)$ the basic p -th Betti number of M , see [7]. Since $0 \leq \dim(\text{Im } \delta_p^b) \leq b_b^p(M)$, for $p \geq n+2$ we have the following result:

Proposition 2.3. *Let M be a K -contact manifold of dimension $2n+1$ such that $H_b^p(M)$ are finite dimensional. Then the coeffective basic de Rham cohomology group $H^p(\mathcal{A}_b(M))$ has finite dimension, for $p \geq n+1$.*

Thus, we can define the *coeffective basic numbers* of M by $c_b^p(M) = \dim H^p(\mathcal{A}_b(M))$, $p \geq n+1$. Notice that $c_b^p(M) = 0$ for $p \leq n-1$.

From (2.11), we have

$$\dim(\operatorname{Im} \delta_{p+1}^b) - \dim H^p(\mathcal{A}_b(M)) + \dim H_b^p(M) - \dim H_b^{p+2}(M) + \dim(\delta_{p+2}^b) = 0,$$

for $p \geq n+1$, from which we deduce

$$\dim(\operatorname{Im} \delta_{p+1}^b) - c_b^p(M) + b_b^p(M) - b_b^{p+2}(M) + \dim(\operatorname{Im} \delta_{p+2}^b) = 0. \quad (2.12)$$

Now, as a consequence of (2.12), we obtain that the coeffective basic numbers of M are bounded by upper and lower limits depending on the basic Betti numbers of the K -contact manifold M .

Theorem 2.3. *Let M be a K -contact manifold of dimension $2n+1$ such that $H_b^p(M)$ are finite dimensional. Then*

$$b_b^p(M) - b_b^{p+2}(M) \leq c_b^p(M) \leq b_b^p(M) + b_b^{p+1}(M) \quad (2.13)$$

for every $p \geq n+1$.

Since $b_b^{2n}(M) = 1$ and $b_b^p(M) = 0$ for every $p \geq 2n+1$ we obtain

Corollary 2.1. *Let M be a K -contact manifold of dimension $2n+1$. Then $c_b^{2n}(M) = 1$.*

We also have

Theorem 2.4. *Let M be a compact Sasakian manifold of dimension $2n+1$. Then*

$$c_b^p(M) = b_b^p(M) - b_b^{p+2}(M), \quad \forall p \geq n+1. \quad (2.14)$$

Proof. The proof follows in a similar manner to the proof of Theorem 5.1. from [9] or Theorem 4.1 from [14] and consist in computing the connecting mapping δ_{p+2}^b .

Let $a \in H_b^{p+2}(M)$. Taking into account the Hodge theory for basic forms on compact Sasakian manifolds, see [7], we may consider the unique harmonic representative φ of the basic de Rham cohomology class a .

Then, by Lemma 2.2, there exists a harmonic basic p -form ψ such that $L\psi = \varphi$. The theorem follows by the definition of the connecting homomorphism, $\delta_{p+2}^b \varphi = [d_b \psi] = 0$. \square

In the end of this section we give a relation between the coeffective de Rham cohomology $H^\bullet(\mathcal{A}(M))$ of a compact K -contact manifold M , [14], and our basic coeffective de Rham cohomology of M .

Recall that if M is compact, the Lie group of isometries of the metric g is compact and then the closure of the subgroup $\{\exp(t\xi)\}_{t \in \mathbb{R}}$ is a compact abelian Lie group, that is it is isomorphic to a torus \mathcal{T} . Denoting by $F_b^\bullet(M)^\mathcal{T}$ the complex of \mathcal{T} -invariant forms on M , then according to Proposition 7.2.1 from [7] the following sequence

$$0 \longrightarrow F_b^\bullet(M) \xrightarrow{i} F^\bullet(M)^\mathcal{T} \xrightarrow{i_\xi} F_b^{\bullet-1}(M) \longrightarrow 0 \quad (2.15)$$

is an exact sequence of complexes which leads to the following long exact sequence in cohomology

$$\dots \longrightarrow H_b^p(M) \xrightarrow{i^*} H^p(M) \xrightarrow{j_p} H_b^{p-1}(M) \xrightarrow{\delta_p} H_b^{p+1}(M) \longrightarrow \dots, \quad (2.16)$$

where δ_p is the connecting homomorphism given by $\delta_p[\varphi] = [L\varphi] = [d\eta] \cup [\varphi]$, and j_p is the composition of the map induced by ι_ξ with the isomorphism $H^p(F^\bullet(M)^\mathcal{T}) \cong H^p(M)$.

Taking into account that $\iota_\xi L = L\iota_\xi$ then

$$0 \longrightarrow \mathcal{A}_b^\bullet(M) \xrightarrow{\iota} \mathcal{A}^\bullet(M)^\mathcal{T} \xrightarrow{\iota_\xi} \mathcal{A}_b^{\bullet-1}(M) \longrightarrow 0 \quad (2.17)$$

is an exact sequence of coeffective complexes, where $\mathcal{A}^\bullet(M)$ is the space of coeffective forms on M , that is $\varphi \in F^\bullet(M)$ such that $L\varphi = 0$ and $\mathcal{A}^\bullet(M)^\mathcal{T}$ is the space of coeffective \mathcal{T} -invariant forms.

Now, if we consider the long exact sequence in cohomology (2.16) for coeffective forms we obtain that the connecting homomorphism δ_p vanish for every p , so we get the short exact sequence in coeffective cohomology,

$$0 \longrightarrow H^p(\mathcal{A}(M)) \xrightarrow{j_p} H^{p-1}(\mathcal{A}_b(M)) \longrightarrow 0, \quad (2.18)$$

for every $p \geq 1$, which say that

Theorem 2.5. *If M is a compact K -contact manifold of dimension $2n+1$, then*

$$H^p(\mathcal{A}(M)) \cong H^{p-1}(\mathcal{A}_b(M)), \quad \forall p = 1, \dots, 2n+1. \quad (2.19)$$

3 Coeffective basic Dolbeault cohomology

In this section we extend our study for basic forms with complex values on a Sasakian manifold M obtaining a coeffective basic Dolbeault cohomology on M . In the case when M is a compact Sasakian manifold, we prove a Hodge decomposition theorem for coeffective basic de Rham cohomology of M , relating this cohomology with basic coeffective Dolbeault cohomology of M . The notions are introduced in a similar manner as for Kähler manifolds, see [18].

For our purpose the complex valued forms on Sasakian manifolds play an important role. For this reason we have need to recall some notions about Dolbeault basic operators on Sasakian manifolds. Notice that endomorphism F determines a complex structure on the contact distribution $\mathcal{D} = \ker \eta$ and on a Sasakian manifold we have $N_F(X, Y) = 0$ for any $X, Y \in \mathcal{D}$, where N_F denotes the Nijenhuis tensor associated to F . Then the complexified of the space of basic p -forms admits the decomposition

$$F_b^p(M) \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{r+s=p} F_b^{r,s}(M), \quad (3.1)$$

where $F_b^{r,s}(M)$ is the space of *basic forms of type (r, s)* , that is the basic forms which can be nonzero only when act on r vector fields from $\mathcal{D}^{1,0}$ and on s vector fields from $\mathcal{D}^{0,1}$. Here we have considered the decomposition of the complexified contact distribution, namely $\mathcal{D} \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{D}^{1,0} \oplus \mathcal{D}^{0,1}$. Then, by applying the classical method used in the case of almost complex manifolds (see for instance [20] pg. 125-126), a simple calculation proves that

$$d_b F_b^{r,s}(M) \subset F_b^{r+1,s}(M) \oplus F_b^{r,s+1}(M),$$

and so the basic exterior derivative admits the decomposition $d_b = \partial_b + \bar{\partial}_b$, where

$$\partial_b : F_b^{r,s}(M) \rightarrow F_b^{r+1,s}(M); \quad \bar{\partial}_b : F_b^{r,s}(M) \rightarrow F_b^{r,s+1}(M).$$

By $d_b^2 = 0$ we deduce

$$\partial_b^2 = \bar{\partial}_b^2 = \partial_b \bar{\partial}_b + \bar{\partial}_b \partial_b = 0. \quad (3.2)$$

On the other hand, we have the decomposition $d_b^* \omega = \partial_b^* \omega + \bar{\partial}_b^* \omega$, induced by the decomposition $d_b = \partial_b + \bar{\partial}_b$ of the basic differential and some formulas similar to (3.2), namely

$$\partial_b^{*2} = \bar{\partial}_b^{*2} = \partial_b^* \bar{\partial}_b^* + \bar{\partial}_b^* \partial_b^* = 0. \quad (3.3)$$

Notice that the classical Hodge identities from Kähler geometry also hold on a compact Sasakian manifold, as shown in [25]. See also Lemma 7.2.7 from [7] or Lemme 3.4.4 from [11] in a more general case of transversally Kählerian foliations. If we define

$$\Delta_b = d_b d_b^* + d_b^* d_b, \Delta_{\partial_b} = \partial_b \partial_b^* + \partial_b^* \partial_b, \Delta_{\bar{\partial}_b} = \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b, \quad (3.4)$$

then we have

Lemma 3.1. ([7]) *On a compact Sasakian manifold one has*

$$\Delta_b = \Delta_{\partial_b} + \Delta_{\bar{\partial}_b} = 2\Delta_{\partial_b} = 2\Delta_{\bar{\partial}_b}.$$

Also the equality $\bar{\partial}_b^2 = 0$ induces the differential complex $(F_b^{r,\bullet}(M), \bar{\partial}_b)$; its cohomology groups

$$H_b^{r,s}(M) = \ker\{F_b^{r,s}(M) \xrightarrow{\bar{\partial}_b} F_b^{r,s+1}(M)\} / \bar{\partial}_b(F_b^{r,s-1}(M)),$$

are the analogous of Dolbeault cohomology groups from Kähler geometry and are called the *basic Dolbeault cohomology groups* of the Sasakian manifold M , [7]. In particular, there is a transverse Hodge theory for the operator $\bar{\partial}_b$, see [7, 12].

Since $d\eta \in F_b^{1,1}(M)$, as in the previous subsection we consider the subspace $\mathcal{A}_b^{r,s}(M) \subset F_b^{r,s}(M)$ defined by

$$\mathcal{A}_b^{r,s}(M) = \{\varphi \in F_b^{r,s}(M) \mid \varphi \wedge d\eta = 0\} = \ker L|_{F_b^{r,s}(M)}.$$

A basic form $\varphi \in \mathcal{A}_b^{r,s}(M)$ is said to be a *coeffective (bigraduate) basic form* of bidegree (r, s) .

From $\bar{\partial}_b d\eta = 0$ the operator L commutes with the operator $\bar{\partial}_b$. Therefore, we can consider the subcomplex of basic Dolbeault complex of M , namely $(\mathcal{A}_b^{r,\bullet}, \bar{\partial}_b)$ for $0 \leq r \leq n$; it is called the *coeffective basic Dolbeault complex* of M . The cohomology groups of this subcomplex are called *coeffective basic Dolbeault cohomology groups* of M and they are denoted by $H^{r,s}(\mathcal{A}_b(M))$.

Taking into account the decomposition (3.1), we obtain the following version of Lemma 2.1, when L acts on $F_b^{r,s}(M)$:

Lemma 3.2. *The operator $L : F_b^{r,s}(M) \rightarrow F_b^{r+1,s+1}(M)$ defined by $L\varphi = \varphi \wedge d\eta$ is injective for $r + s \leq n - 1$ and surjective for $r + s \geq n - 1$.*

As a consequence of Lemma 3.2, one gets

Proposition 3.1. *Let M be a regular Sasakian manifold of dimension $2n+1$. Then $\mathcal{A}_b^{r,s}(M) = \{0\}$ for $r + s \leq n - 1$, therefore*

$$H^{r,s}(\mathcal{A}_b(M)) = \{0\}, \text{ for } r + s \leq n - 1. \quad (3.5)$$

Let us denote by $[d\eta]_D$ the basic Dolbeault class of $d\eta$ in $H_b^{1,1}(M)$ and we consider the subspace of $H_b^{r,s}(M)$ given by the basic Dolbeault cohomology classes truncated by the class $[d\eta]_D$, namely,

$$\tilde{H}_b^{r,s}(M) = \{a \in H_b^{r,s}(M) \mid a \wedge [d\eta]_D = 0\}. \quad (3.6)$$

Next we define the mapping $\alpha_{r,s} : H^{r,s}(\mathcal{A}_b(M)) \rightarrow \tilde{H}_b^{r,s}(M)$ by

$$\alpha_{r,s}(\{\varphi\}_D) = [\varphi]_D, \quad (3.7)$$

where $\{\varphi\}_D$ denotes the cohomology class of a coeffective basic form φ in $H^{r,s}(\mathcal{A}_b(M))$ and $[\varphi]_D$ denotes the cohomology class of a basic form φ in $H_b^{r,s}(M)$. This mapping permits us to give a relation between the coeffective basic Dolbeault cohomology groups of the Sasakian manifold M and the subspaces of the basic Dolbeault cohomology groups given by (3.6), namely

Proposition 3.2. *If M is a regular Sasakian manifold of dimension $2n + 1$, the mapping $\alpha_{r,s}$ defined by (3.7) is surjective for $r + s \geq n$.*

Proof. It follows in a similar manner with the proof of Proposition 2.6 from [18] (for Kähler manifolds) using the same technique as in Proposition 2.2. \square

In the following, we relate the coeffective basic Dolbeault cohomology groups and the subspaces of the basic Dolbeault cohomology groups given by (3.6) for compact Sasakian manifolds and we prove a coeffective version of the basic Hodge decomposition theorem for coeffective basic Dolbeault cohomology.

Now by Lemma 3.1 and taking into account that $\Delta_{\bar{\partial}_b}$ preserves the bigraduation of basic forms, we have the following version of Lemma 2.2 when L acts on the space $\mathcal{H}_b^{r,s}(M) = \ker \Delta_{\bar{\partial}_b}$ of harmonic basic forms of type (r, s) :

Lemma 3.3. *The operator $L : \mathcal{H}_b^{s-1,s-1}(M) \rightarrow \mathcal{H}_b^{r,s}(M)$ is surjective for $r + s \geq n + 1$.*

Theorem 3.1. *For a compact Sasakian manifold M of dimension $2n + 1$, we have*

$$H^{r,s}(\mathcal{A}_b(M)) \cong \tilde{H}_b^{r,s}(M), \quad (3.8)$$

for $r + s \neq n$.

Proof. It follows in a similar manner with the proof of Theorem 3.2 from [18] (for compact Kähler manifolds), using the same technique as in Theorem 2.2. \square

Now, using the above result, by similar arguments as in the proof of Theorem 3.3 from [18] (for Kähler manifolds) we will obtain a Hodge decomposition theorem for coeffective basic Dolbeault cohomology of compact Sasakian manifolds.

Theorem 3.2. *If M is a compact Sasakian manifold of dimension $2n + 1$ then we have*

$$i) \quad \tilde{H}^p(M) \cong \bigoplus_{r+s}^p \tilde{H}_b^{r,s}(M).$$

$$ii) \quad H^p(\mathcal{A}_b(M)) \cong \bigoplus_{r+s}^p H^{r,s}(\mathcal{A}_b(M)) \text{ for } r + s \geq n + 1.$$

Proof. Let $a \in \tilde{H}_b^p(M)$ and φ a representative of a . Without loss the generality we can assume that φ is basic harmonic. From (3.1) we have the decomposition

$$\varphi = \varphi_{p,0} + \dots + \varphi_{r,s} + \dots + \varphi_{0,p},$$

and taking into account that $\ker \Delta_b = \ker \Delta_{\bar{\partial}_b}$, from $\Delta_{\bar{\partial}_b} \varphi = \Delta_b \varphi = 0$ and since $\Delta_{\bar{\partial}_b}$ preserves the bigraduation, we have

$$\Delta_{\bar{\partial}_b} \varphi_{p,0} = \dots = \Delta_{\bar{\partial}_b} \varphi_{r,s} = \dots = \Delta_{\bar{\partial}_b} \varphi_{0,p} = 0.$$

Moreover, since $d\eta$ is of bidegree $(1,1)$ basic form and $\varphi \wedge d\eta = 0$, we have

$$\varphi_{p,0} \wedge d\eta = \dots = \varphi_{r,s} \wedge d\eta = \dots = \varphi_{0,p} \wedge d\eta = 0.$$

Taking into account the Hodge theory for basic forms on Sasakian manifolds, see [7], we have

$$\tilde{H}_b^{r,s}(M) \cong \left\{ \varphi \in \mathcal{H}_b^{r,s}(M) \mid \varphi \wedge d\eta \in \bar{\partial}_b \left(F_b^{r+1,s}(M) \right) \right\} \cong \{ \varphi \in \mathcal{H}_b^{r,s}(M) \mid \varphi \wedge d\eta = 0 \}. \quad (3.9)$$

Thus, part i) follows by (3.9).

Now, from part i), Theorem 2.2 and Theorem 3.1, we have

$$H^p(\mathcal{A}_b(M)) \cong \tilde{H}_b^p(M) \cong \bigoplus_{r+s=p} \tilde{H}_b^{r,s}(M) \cong \bigoplus_{r+s=p} H^{r,s}(\mathcal{A}_b(M)),$$

and it follows part ii). □

Let us denote by $c_b^{r,s}(M) = \dim H^{r,s}(\mathcal{A}_b(M))$.

Corollary 3.1. *For a compact Sasakian manifold of dimension $2n+1$ we have*

$$c_b^p(M) = \sum_{r+s=p} c_b^{r,s}(M),$$

for $p \geq n+1$.

Remark 3.1. Using the same technique as in the previous subsection we can relate the coeffective basic Dolbeault cohomology of Sasakian manifolds by means of a long exact sequence in basic cohomology and we can prove that

$$h_b^{r,s}(M) - h_b^{r+1,s+1}(M) \leq c_b^{r,s}(M) \leq h_b^{r,s}(M) + h_b^{r,s+1}(M), \quad \forall r+s \geq n+1,$$

where $h_b^{r,s}(M) = \dim H_b^{r,s}(M)$ are the basic Hodge (r,s) -numbers of M .

Also, when M is compact we obtain

$$c_b^{r,s}(M) = h_b^{r,s}(M) - h_b^{r+1,s+1}(M), \quad \forall r+s \geq n+1.$$

4 Coeffective basic Bott-Chern cohomology

In this section we firstly define basic Bott-Chern and Aeppli cohomology of a Sasakian manifold M and we obtain a Hodge-Bott-Chern decomposition theorem for basic forms of M . Next, in similar manner with the study of coeffective basic de Rham and Dolbeault cohomology of M , we define and study a coeffective Bott-Chern cohomology for basic forms on M .

4.1 Hodge-Bott-Chern decomposition for basic forms

In the first part of this subsection, we define the basic Bott-Chern and Aeppli cohomology groups of M . In the second part we define a basic Bott-Chern Laplacian and we obtain a Hodge-Bott-Chern type decomposition theorem for basic forms on M .

Definition 4.1. The differential complex

$$\dots \longrightarrow F_b^{r-1,s-1}(M) \xrightarrow{\partial_b \bar{\partial}_b} F_b^{r,s}(M) \xrightarrow{\partial_b \oplus \bar{\partial}_b} F_b^{r+1,s}(M) \oplus F_b^{r,s+1}(M) \longrightarrow \dots \quad (4.1)$$

is called the *basic Bott-Chern complex* of M and the *basic Bott-Chern cohomology groups* of M of bidegree (r, s) , are given by

$$H_{b,BC}^{r,s}(M) = \frac{\ker\{\partial_b : F_b^{r,s}(M) \rightarrow F_b^{r+1,s}(M)\} \cap \ker\{\bar{\partial}_b : F_b^{r,s}(M) \rightarrow F_b^{r,s+1}(M)\}}{\text{Im}\{\partial_b \bar{\partial}_b : F_b^{r-1,s-1}(M) \rightarrow F_b^{r,s}(M)\}}.$$

Next, we consider the dual of the basic Bott-Chern cohomology groups, given by

$$H_{b,A}^{r,s}(M) = \frac{\ker\{\partial_b \bar{\partial}_b : F_b^{r,s}(M) \rightarrow F_b^{r+1,s+1}(M)\}}{\text{Im}\{\partial_b : F_b^{r-1,s}(M) \rightarrow F_b^{r,s}(M)\} + \text{Im}\{\bar{\partial}_b : F_b^{r,s-1}(M) \rightarrow F_b^{r,s}(M)\}}$$

called the *basic Aeppli cohomology groups* of bidegree (r, s) of M .

Proposition 4.1. *The exterior product induces a bilinear map*

$$\wedge : H_{b,BC}^{p,q}(M) \times H_{b,A}^{r,s}(M) \rightarrow H_{b,A}^{p+r,q+s}(M). \quad (4.2)$$

Proof. Let $\varphi, \psi \in F_b^{r,s}(M)$. If φ is d_b -closed and ψ is $\partial_b \bar{\partial}_b$ -closed then $\varphi \wedge \psi$ is $\partial_b \bar{\partial}_b$ -closed. Also, if φ is d_b -closed and ψ is d_b -exact then $\varphi \wedge \psi$ is d_b -exact and if φ is $\partial_b \bar{\partial}_b$ -exact and ψ is $\partial_b \bar{\partial}_b$ -closed then $\varphi \wedge \psi$ is d_b -exact.

For the last assertion, we have

$$\varphi \wedge \psi = \partial_b \bar{\partial}_b \theta \wedge \psi = \frac{1}{2} d_b [(\bar{\partial}_b - \partial_b) \theta \wedge \psi + (-1)^{r+s} \theta \wedge (\partial_b - \bar{\partial}_b) \psi]. \quad \square$$

In particular,

$$H_{b,BC}^{r,s}(M) \times H_{b,A}^{n-r,n-s}(M) \rightarrow H_{b,A}^{n,n}(M) = H_b^{2n}(M) \cong \mathbb{R}.$$

In the following, we define the *Bott-Chern Laplacian* for basic forms of type (r, s) by

$$\Delta_{BC}^b = \partial_b \bar{\partial}_b (\partial_b \bar{\partial}_b)^* + \partial_b^* \partial_b + \bar{\partial}_b^* \bar{\partial}_b. \quad (4.3)$$

This operator is self-adjoint, i.e. $\langle \Delta_{BC}^b \varphi, \psi \rangle_b = \langle \varphi, \Delta_{BC}^b \psi \rangle_b$. Also, for a basic form $\varphi \in F_b^{r,s}(M)$ we have

$$\begin{aligned} \langle \Delta_{BC}^b \varphi, \varphi \rangle_b &= \langle \partial_b \bar{\partial}_b (\partial_b \bar{\partial}_b)^* \varphi + \partial_b^* \partial_b \varphi + \bar{\partial}_b^* \bar{\partial}_b \varphi, \varphi \rangle_b \\ &= \langle (\partial_b \bar{\partial}_b)^* \varphi, (\partial_b \bar{\partial}_b)^* \varphi \rangle_b + \langle \partial_b \varphi, \partial_b \varphi \rangle_b + \langle \bar{\partial}_b \varphi, \bar{\partial}_b \varphi \rangle_b \\ &= ||(\partial_b \bar{\partial}_b)^* \varphi||^2 + ||\partial_b \varphi||^2 + ||\bar{\partial}_b \varphi||^2 \end{aligned}$$

where $||\varphi||^2 = \langle \varphi, \varphi \rangle_b$. Thus, we obtain

Proposition 4.2. $\Delta_{BC}^b \varphi = 0$ if and only if $(\partial_b \bar{\partial}_b)^* \varphi = \partial_b \varphi = \bar{\partial}_b \varphi = 0$.

We denote by $\mathcal{H}_{b,BC}^{r,s}(M)$ the space of Δ_{BC}^b -harmonic basic forms of type (r, s) on M .

Following the same ideas from [27], we now show that $H_{b,BC}^{*,*}(M)$ is finite dimensional by analyzing the space of its harmonic basic forms. Firstly, let us consider a related fourth-order differential operator which is elliptic (see [12] for general transversally Hermitian foliations), namely

$$\tilde{\Delta}_{BC}^b = \partial_b \bar{\partial}_b \partial_b^* \bar{\partial}_b^* + \bar{\partial}_b^* \partial_b^* \partial_b \bar{\partial}_b + \bar{\partial}_b^* \partial_b \partial_b^* \bar{\partial}_b + \partial_b^* \bar{\partial}_b \bar{\partial}_b^* \partial_b + \bar{\partial}_b^* \bar{\partial}_b + \partial_b^* \partial_b. \quad (4.4)$$

This operator has the same kernel as Δ_{BC}^b . Indeed

$$0 = \langle \varphi, \tilde{\Delta}_{BC}^b \varphi \rangle_b = \|\partial_b \varphi\|^2 + \|\bar{\partial}_b \varphi\|^2 + \|(\partial_b \bar{\partial}_b)^* \varphi\|^2 + \|\partial_b \bar{\partial}_b \varphi\|^2 + \|\partial_b^* \bar{\partial}_b \varphi\|^2 + \|\bar{\partial}_b^* \partial_b \varphi\|^2$$

and the three additional terms clearly do not give any additional conditions and are automatically zero by requirement $\partial_b \varphi = \bar{\partial}_b \varphi = 0$. Essentially, the presence of the second-order differential terms ensures that the spaces $\ker \Delta_{BC}^b$ and $\ker \tilde{\Delta}_{BC}^b$ coincides. Using the classical Hodge identities for Sasakian manifolds, (see Lemma 7.2.7 from [7]), in relation (4.4), we also obtain:

Proposition 4.3. *If M is a compact Sasakian manifold of dimension $2n + 1$, then*

$$\tilde{\Delta}_{BC}^b = \Delta_{\bar{\partial}_b} \Delta_{\partial_b} + \partial_b^* \partial_b + \bar{\partial}_b^* \bar{\partial}_b.$$

Moreover, the harmonic spaces $\mathcal{H}_b^{r+s}(M) \cap F_b^{r,s}(M)$, $\mathcal{H}_b^{r,s}(M)$ and $\mathcal{H}_{b,BC}^{r,s}(M)$ coincides, and also $d\eta$ is harmonic basic $(1, 1)$ -form with respect to every Laplacian: Δ_b , $\Delta_{\bar{\partial}_b}$ and Δ_{BC}^b , respectively.

We have now

Theorem 4.1. *Let M be a compact Sasakian manifold of dimension $2n + 1$. Then*

- (i) $\dim \mathcal{H}_{b,BC}^{r,s}(M) < \infty$;
- (ii) *There is an orthogonal decomposition*

$$F_b^{r,s}(M) = \mathcal{H}_{b,BC}^{r,s}(M) \oplus \text{Im}(\partial_b \bar{\partial}_b) \oplus (\text{Im} \partial_b^* + \text{Im} \bar{\partial}_b^*); \quad (4.5)$$

- (iii) *There are the canonical isomorphisms:*

$$\mathcal{H}_{b,BC}^{r,s}(M) \cong H_{b,BC}^{r,s}(M) \cong H_b^{r,s}(M).$$

Proof. (i) Because only the highest order differential need to be kept for computing the principal symbol of a Laplace operator, by the calculations of $\tilde{\Delta}_{BC}^b$ from Proposition 4.3, it follows that the principal symbol of $\tilde{\Delta}_{BC}^b$ is equal to that of the square of the operator $\Delta_{\bar{\partial}_b}$, so it is positive. Thus $\tilde{\Delta}_{BC}^b$ is elliptic and hence its kernel, $\mathcal{H}_{b,BC}^{r,s}(M)$, is finite dimensional.

With $\tilde{\Delta}_{BC}^b$ elliptic, assertion (ii) then follows directly by applying elliptic theory. For (iii), using the decomposition of (ii), we have

$$\ker(\partial_b + \bar{\partial}_b) = \mathcal{H}_{b,BC}^{r,s}(M) \oplus \text{Im}(\partial_b \bar{\partial}_b). \quad (4.6)$$

This must be so since for a form $\varphi \in F_b^{r,s}(M)$ given by $\varphi = \psi + \partial_b \bar{\partial}_b \theta + \partial_b^* \theta_1 + \bar{\partial}_b^* \theta_2$, where $\psi \in \mathcal{H}_{b,BC}^{r,s}(M)$, we have $\partial_b \varphi = \bar{\partial}_b \varphi = 0$ if and only if

$$\begin{aligned} 0 &= \langle \theta_1, \partial_b(\partial_b^* \theta_1 + \bar{\partial}_b^* \theta_2) \rangle_b + \langle \theta_2, \bar{\partial}_b(\partial_b^* \theta_1 + \bar{\partial}_b^* \theta_2) \rangle_b \\ &= \langle \partial_b^* \theta_1 + \bar{\partial}_b^* \theta_2, \partial_b^* \theta_1 + \bar{\partial}_b^* \theta_2 \rangle_b \\ &= \|\partial_b^* \theta_1 + \bar{\partial}_b^* \theta_2\|^2 \end{aligned}$$

which imply $\partial_b^* \theta_1 + \bar{\partial}_b^* \theta_2 = 0$, i.e. desired decomposition from (4.6). Thus every cohomology class of $H_{b,BC}^{\bullet,\bullet}(M)$ contains a unique harmonic representative and $\mathcal{H}_{b,BC}^{r,s}(M) \cong H_{b,BC}^{r,s}(M)$, i.e. the first isomorphism of (iii). Since $\ker \tilde{\Delta}_{BC}^b = \ker \Delta_{\bar{\partial}_b}$, the second isomorphism of (iii) it follows by $H_b^{r,s}(M) \cong \mathcal{H}_b^{r,s}(M)$ and $\mathcal{H}_b^{r,s}(M) = \mathcal{H}_{b,BC}^{r,s}(M)$. \square

Corollary 4.1. *If M is a compact Sasakian manifold of dimension $2n + 1$, then $H_{b,BC}^{r,s}(M)$ is finite dimensional.*

Now, let us define the *Aeppli Laplacian* for basic forms of type (r, s) on M by

$$\Delta_A^b + \partial_b \partial_b^* + \bar{\partial}_b \bar{\partial}_b^* + (\partial_b \bar{\partial}_b)^* \partial_b \bar{\partial}_b \quad (4.7)$$

which is not elliptic, but if we change it by

$$\tilde{\Delta}_A^b = \partial_b \partial_b^* + \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \partial_b \bar{\partial}_b + \partial_b \bar{\partial}_b \bar{\partial}_b^* \partial_b^* + \partial_b \bar{\partial}_b^* \bar{\partial}_b \partial_b^* + \bar{\partial}_b \partial_b^* \partial_b \bar{\partial}_b^* \quad (4.8)$$

this is elliptic.

Now, if we denote $\mathcal{H}_{b,A}^{r,s}(M) = \ker \tilde{\Delta}_A^b \cap F_b^{r,s}(M)$, then by applying elliptic theory arguments, similar to Theorem 4.1, we have

Theorem 4.2. *Let M be a compact Sasakian manifold of dimension $2n + 1$. Then*

- (i) $\dim \mathcal{H}_{b,A}^{r,s}(M) < \infty$;
- (ii) *There is an orthogonal decomposition*

$$F_b^{r,s}(M) = \mathcal{H}_{b,A}^{r,s}(M) \oplus (\text{Im} \partial_b + \text{Im} \bar{\partial}_b) \oplus \text{Im}(\bar{\partial}_b^* \partial_b^*); \quad (4.9)$$

- (iii) *There is a canonical isomorphism:*

$$\mathcal{H}_{b,A}^{r,s}(M) \cong H_{b,A}^{r,s}(M).$$

Corollary 4.2. *If M is a compact Sasakian manifold, then $H_{b,A}^{r,s}(M)$ is finite dimensional.*

Finally, let us remark that \star_b gives an isomorphism $H_{b,BC}^{r,s}(M) \approx H_{b,A}^{n-r,n-s}(M)$.

4.2 Coeffective Bott-Chern cohomology for basic forms

In this subsection we define and study a coeffective Bott-Chern cohomology for basic forms on Sasakian manifolds.

Since the operator L commutes with both operators ∂_b and $\bar{\partial}_b$, we can consider the subcomplex of Bott-Chern complex of M

$$\dots \longrightarrow \mathcal{A}_b^{r-1,s-1}(M) \xrightarrow{\partial_b \bar{\partial}_b} \mathcal{A}_b^{r,s}(M) \xrightarrow{\partial_b \oplus \bar{\partial}_b} \mathcal{A}_b^{r+1,s}(M) \oplus \mathcal{A}_b^{r,s+1}(M) \longrightarrow \dots \quad (4.10)$$

for $1 \leq r, s \leq n$; called the *coeffective basic Bott-Chern complex* of M . The cohomology groups of the complex (4.10) are called *coeffective basic Bott-Chern cohomology groups* of M and they are denoted by $H_{BC}^{r,s}(\mathcal{A}_b(M))$.

By Lemma 3.2, one gets

Proposition 4.4. *Let M be a regular Sasakian manifold of dimension $2n + 1$. Then*

$$H_{BC}^{r,s}(\mathcal{A}_b(M)) = \{0\}, \text{ for } r + s \leq n - 1. \quad (4.11)$$

Since $\partial_b d\eta = \bar{\partial}_b d\eta = 0$, we have that $[d\eta]_{BC} \in H_{BC}^{1,1}(M)$ and we consider the subspace of $H_{b,BC}^{r,s}(M)$ given by the basic Bott-Chern cohomology classes truncated by the basic Bott-Chern class $[d\eta]_{BC}$, namely,

$$\tilde{H}_{b,BC}^{r,s}(M) = \{a \in H_{b,BC}^{r,s}(M) \mid a \wedge [d\eta]_{BC} = 0\}. \quad (4.12)$$

Next we define the mapping $\alpha_{r,s} : H_{BC}^{r,s}(\mathcal{A}_b(M)) \rightarrow \tilde{H}_{b,BC}^{r,s}(M)$ by

$$\alpha_{r,s}(\{\varphi\}_{BC}) = [\varphi]_{BC}, \quad (4.13)$$

where $\{\varphi\}_{BC}$ denotes the cohomology class of a coeffective basic form φ in $H_{BC}^{r,s}(\mathcal{A}_b(M))$ and $[\varphi]_{BC}$ denotes the basic cohomology class of a basic form φ in $H_{b,BC}^{r,s}(M)$. This mapping permits us to give a relation between the coeffective basic Bott-Chern cohomology groups of the Sasakian manifold M and the subspaces of the basic Bott-Chern cohomology groups given by (4.12), just in the case of coeffective basic de Rham and Dolbeault cohomology of M . In the following, our aim is to find a link between the coeffective basic Bott-Chern cohomology groups and the subspaces of the basic Bott-Chern cohomology groups given by (4.12) for compact Sasakian manifolds and to prove a coeffective version of the basic Hodge decomposition theorem for basic Bott-Chern cohomology.

Proposition 4.5. *If M is a regular Sasakian manifold of dimension $2n + 1$, the mapping $\alpha_{r,s}$ defined by (4.13) is surjective for $r + s \geq n$.*

Proof. Let $a \in \tilde{H}_{b,BC}^{r,s}(M)$, that is, $a \in H_{b,BC}^{r,s}(M)$ and $a \wedge [d\eta]_{BC} = 0$ in $H_{b,BC}^{r+1,s+1}(M)$. Consider a representative φ of a and suppose that $\varphi \notin \mathcal{A}_b^{r,s}(M)$ (notice that if $\varphi \in \mathcal{A}_b^{r,s}(M)$, then φ defines a basic cohomology class in $H_{BC}^{r,s}(\mathcal{A}_b(M))$ such that $\alpha(\{\varphi\}_{BC}) = a$).

Since $a \wedge [d\eta]_{BC} = 0$, then there exists $\sigma \in F_b^{r,s}(M)$ such that $\varphi \wedge d\eta = \partial_b \bar{\partial}_b \sigma$. Then, from Lemma 3.2, there exists $\gamma \in F_b^{r-1,s-1}(M)$ such that $L\gamma = \sigma$. Thus, $L(\varphi - \partial_b \bar{\partial}_b \gamma) = 0$ and $\partial_b(\varphi - \partial_b \bar{\partial}_b \gamma) = \bar{\partial}_b(\varphi - \partial_b \bar{\partial}_b \gamma) = 0$. Hence, $\varphi - \partial_b \bar{\partial}_b \gamma$ defines a basic cohomology class in $H_{BC}^{r,s}(\mathcal{A}_b(M))$ such that $\alpha_{r,s}(\{\varphi - \partial_b \bar{\partial}_b \gamma\}_{BC}) = a$. \square

Now, taking into account the relation (2.4), the classical Hodge identities for Sasakian manifolds and Proposition 4.3, we have

$$\tilde{\Delta}_{BC}^b L - L \tilde{\Delta}_{BC}^b = -2i \partial_b \bar{\partial}_b, \quad (4.14)$$

so, if $\varphi \in \mathcal{H}_{b,BC}^{r,s}(M)$ then $L\varphi \in \mathcal{H}_{b,BC}^{r+1,s+1}(M)$.

Theorem 4.3. *For a compact Sasakian manifold M of dimension $2n + 1$, we have*

$$H_{BC}^{r,s}(\mathcal{A}_b(M)) \cong \tilde{H}_{b,BC}^{r,s}(M), \quad (4.15)$$

for $r + s \notin \{n, n + 1\}$.

Proof. Using an argument similar to that used in [18] the proof it follows by two cases using the same technique as in Theorem 2.2.

Case 1: $r + s \leq n - 1$.

From (4.11) we know that $H_{BC}^{r,s}(\mathcal{A}_b(M)) = \{0\}$ for $r + s \leq n - 1$. Moreover, from Theorem 4.1 (the first isomorphism of (iii)), we have

$$\tilde{H}_{b,BC}^{r,s}(M) \cong \left\{ \varphi \in \mathcal{H}_{b,BC}^{r,s}(M) \mid \varphi \wedge d\eta \in \partial_b \bar{\partial}_b (F_b^{r,s}(M)) \right\} \cong \left\{ \varphi \in \mathcal{H}_{b,BC}^{r,s}(M) \mid \varphi \wedge d\eta = 0 \right\}. \quad (4.16)$$

Thus, from Lemma 3.2 we conclude that $\tilde{H}_{b,BC}^{r,s}(M) = \{0\}$ for $r + s \leq n - 1$. This finishes the proof for $r + s \leq n - 1$.

Case 2: $r + s \geq n + 2$.

We shall see that the mapping $\alpha_{r,s}$ given by (4.13) is an isomorphism for $r + s \geq n + 1$. From Proposition 4.5, it is sufficient to show the injection.

Let $a \in H_{BC}^{r,s}(\mathcal{A}_b(M))$ such that $\alpha_{r,s}(a) = 0$ in $\tilde{H}_{b,BC}^{r,s}(M)$ and suppose that φ is a representative of a . Since $\alpha_{r,s}(a) = \alpha_{r,s}(\{\varphi\}_{BC}) = [\varphi]_{BC} = 0$ in $\tilde{H}_{b,BC}^{r,s}(M)$, there exists $\psi \in F_b^{r-1,s-1}(M)$ such that

$$\varphi = \partial_b \bar{\partial}_b \psi.$$

Suppose $\psi \notin \mathcal{A}_b^{r-1,s-1}(M)$ (notice that if $\psi \in \mathcal{A}_b^{r-1,s-1}(M)$, then $a = 0$ and we conclude the proof). Since L commutes with ∂_b and $\bar{\partial}_b$, then $\partial_b \bar{\partial}_b (L\psi) = L(\partial_b \bar{\partial}_b \psi) = L\varphi = 0$; therefore $L\psi$ defines a basic Aepli cohomology class $[L\psi]_A \in H_{b,A}^{r,s}(M)$. From the Theorem 4.2 (the isomorphism (iii)), we have

$$L\psi = \psi_1 + \partial_b \gamma_1 + \bar{\partial}_b \gamma_2,$$

for $\psi_1 \in \mathcal{H}_{b,A}^{r,s}(M)$, $\gamma_1 \in F_b^{r-1,s}(M)$ and $\gamma_2 \in F_b^{r,s-1}(M)$. Since $r + s \geq n + 2$ and $\psi_1 \in \mathcal{H}_{b,A}^{r,s}(M) = \mathcal{H}_b^{r,s}(M)$ by Lemma 3.3 there exists $\psi_2 \in \mathcal{H}_b^{r-1,s-1}(M) = \mathcal{H}_{b,A}^{r-1,s-1}(M)$ such that $L\psi_2 = \psi_1$ and since $r + s - 1 \geq n + 1$ by Lemma 3.2 there exist $\sigma_1 \in F_b^{r-2,s-1}(M)$ and $\sigma_2 \in F_b^{r-1,s-2}(M)$ such that $\gamma_1 = L\sigma_1$ and $\gamma_2 = L\sigma_2$, respectively. Thus,

$$L(\psi - \psi_2 - \partial_b \sigma_1 - \bar{\partial}_b \sigma_2) = 0 \text{ and } \partial_b \bar{\partial}_b (\psi - \psi_2 - \partial_b \sigma_1 - \bar{\partial}_b \sigma_2) = \varphi.$$

Then, $a = \{\varphi\}_{BC}$ is the zero basic class in $H_{BC}^{r,s}(\mathcal{A}_b(M))$ and this finishes the proof. \square

Now, using the above result, by similar arguments as in the proof of Theorem 3.2 we obtain a Hodge decomposition theorem for coeffective basic Bott-Chern cohomology of compact Sasakian manifolds.

Theorem 4.4. *If M is a compact Sasakian manifold of dimension $2n + 1$ then we have*

$$i) \quad \tilde{H}^p(M) \cong \bigoplus_{r+s}^p \tilde{H}_{b,BC}^{r,s}(M).$$

$$ii) \quad H^p(\mathcal{A}_b(M)) \cong \bigoplus_{r+s}^p H_{BC}^{r,s}(\mathcal{A}_b(M)) \text{ for } r + s \geq n + 2.$$

Finally, let us denote by $c_{b,BC}^{r,s}(M) = \dim H_{BC}^{r,s}(\mathcal{A}_b(M))$. Then

Corollary 4.3. *If M is a compact Sasakian manifold of dimension $2n + 1$, then*

$$c_b^p(M) = \sum_{r+s=p} c_{b,BC}^{r,s}(M), \text{ for } r + s \geq n + 2.$$

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References

- [1] Aepli, A., *Some exact sequences in cohomology theory for Kähler manifolds*, Pacific J. Math., **12** (3) (1962), 791-799.
- [2] de Andrés, L. C., Fernández, M., de León, M., Ibáñez, R., Mencía, J., *On the coeffective cohomology of compact symplectic manifolds*. C. R. Acad. Sci. Paris, 318, Série I, (1994), 231–236.
- [3] Blair, D. E., *Contact manifolds in Riemannian geometry*. Lecture Notes in Math., **509**, Springer, Berlin, 1976.
- [4] Blair, D. E., *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics, 203. Birkhäuser Boston, Inc., Boston, MA, 2002.
- [5] Bigolin, B. *Gruppi di Aepli*. (Italian) Ann. Scuola Norm. Sup. Pisa (3) **23** (1969), 259–287.
- [6] Bouché, T., *La cohomologie coeffective d’une variété symplectique*. Bull. Sci. Math. **114** (1990) (2) 115-122.
- [7] Boyer C. P., Galicki, K., *Sasakian geometry*, Oxford Mathematical Monographs. Oxford University Press, Oxford, 2008.
- [8] Chinea, D., de León, M., Marrero, J. C., *Topology of cosymplectic manifolds*. J. Math. Pures et Appl., **72** (6) (1993), 567–591.
- [9] Chinea, D., de León, M., Marrero, J. C., *Coeffective cohomology on cosymplectic manifolds*. Bull. Sci. Math. **119** (1995)(1) 3–20.
- [10] Chinea, D., Marrero, J. C., de Leon, M., *A Canonical Differential Complex for Jacobi Manifolds*. Michigan Math. J. **45** (1998) 547–579.
- [11] El Kacimi Alaoui, A., *Opérateurs transversalement elliptiques sur un feuilletage riemannien et applications*, Compositio Math. **73** (1990), 57–106.
- [12] El Kacimi Alaoui, A., Gmira, B., *Stabilité du caractère Kählérien transverse*, Israel J. of Math., **101** (1997), 323–347.
- [13] El Kacimi Alaoui, A., Hector, G., *Décomposition de Hodge basique pour un feuilletage riemannien*, Ann. Inst. Fourier, Grenoble **36** (3) (1986), 207–227.
- [14] Fernández, M., Ibáñez, R., de León, M., *Coeffective and de Rham cohomologies of symplectic manifolds*. J. Geom. Phys. **27** (1998) no 3-4, 281–296.
- [15] Fernández, M., Ibáñez, R., de León, M., *Coeffective and de Rham cohomologies on almost contact manifolds*. Diff. Geom. Appl. **8** (1998), 285–303.
- [16] Fernández, M., Ibáñez, R., de León, M., *Coeffective cohomology of Quaternionic Kähler Manifolds*. J. Szenthe (ed.), New Developments in Differential Geometry, Budapest 1996 Springer Science+Business Media Dordrecht 1999, pp. 111–121.
- [17] Fujitani, T., *Complex-valued differential forms on normal contact Riemannian manifolds*. Tôhoku Math. J., 18(1966), 349–361.

- [18] Ibáñez, R., *Coeffective-Dolbeault cohomology of compact indefinite Kähler manifolds*. Osaka J. Math. **34** (1997), 553–571.
- [19] Ida, C., Pitiş, G., *Basic cohomologies on K-contact manifolds*. Preprint submmited.
- [20] Kobayashi, S., Nomizu, K., *Foundations of differential geometry*. Interscience Publ., vol. I(1963), vol. II (1969).
- [21] Morrow, J., Kodaira, K., *Complex Manifolds*, AMS Chelsea Publ., 1971.
- [22] Ogawa, Y., *On C-harmonic forms in a compact Sasakian space*. *Tôhoku Math. J.*, **19**(1967), 267–296.
- [23] Pitiş, G., *Contact Geometry: Sasaki manifolds, Kenmotsu manifolds*, Manuscript (unpublished).
- [24] Schweitzer, M., *Autour de la cohomologie de Bott-Chern*. Available to arXiv:0709.3528v1[math. AG] 21 Sep 2007 and Prépublications de Inst. Fourier, no. 703 (2007), www-fourier.ujf-grenoble.fr/prepublications.html.
- [25] Tanaka, N., *A differential Geometric Study on Strongly Pseudo-Convex Manifolds* (Lectures in Mathematics 9) Kinokuniya, 1975.
- [26] Tondeur, Ph., *Foliations on Riemannian manifolds*. Universitext, Springer-Verlag, 1988.
- [27] Tseng, L.-S., Yau, S.-T., *Cohomology and Hodge Theory on Symplectic Manifolds: I, II*. J. Differ. Geom. **91** (2012), no. 3, 383–416, 417–444.
- [28] Vaisman, I., *Cohomology and differential forms*, M. Dekker Publ. House, 1973.

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